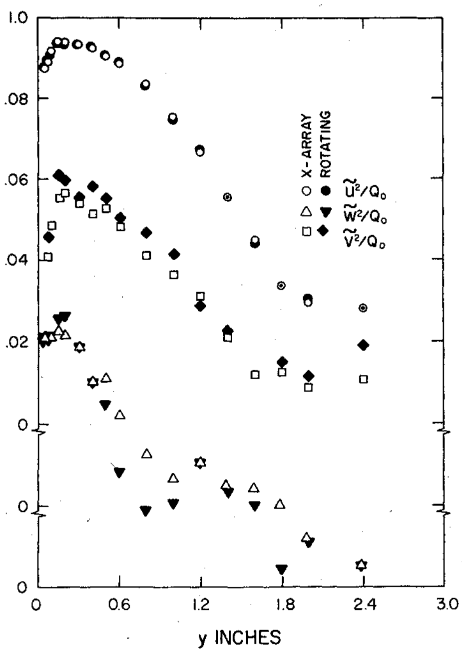

Fig. 3 Nondimensionalized normal Reynolds stresses.

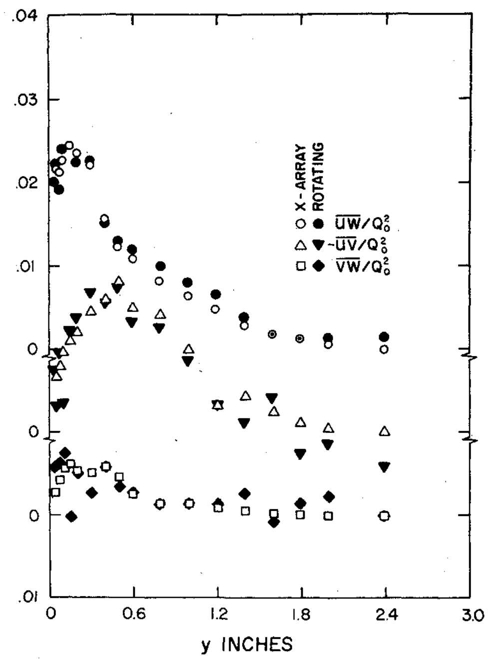

Fig. 4 Nondimensionalized shear Reynolds stresses.

specious stress value, particularly in the slant-sensor output. These typical results strongly suggest the need to obtain a profile of any given stress rather than rely on any single discrete data point. In this rotating-wire method, the horizontal wire gives the $\overline{u^2}$, $\overline{w^2}$, and $\overline{uw}$ stresses. These are seen to have better repeatability than the $\overline{v^2}$, $\overline{uv}$, and $\overline{vw}$ stresses that follow from the slant-wire output, with the earlier three stresses that follow from the slant-wire output, with the earlier three stresses also used as input. The uncertainties in these input stresses necessarily are reflected in the uncertainties of the $\overline{v^2}$, $\overline{uv}$, and $\overline{vw}$ stresses. Various exercises were attempted where all six Reynolds stresses were found using only the rotating-slant sensor output and a set of six linear simultaneous equations. In nearly all cases, there was substantially more scatter and much less confidence in this approach.

Conclusions

A comparison of all six terms of the Reynolds stress tensor measured in a pressure-driven 3DTBL by two independent methods is presented. In the course of two experiments, complete Reynolds stress tensor profiles have been measured independently for eight stations by the x-array method and 14 stations by the rotating-sensor method. Based on this extensive experience, a comparison of overall effort and some generalized comments on typical output can be made.

Agreement between the two methods is judged good. It becomes a matter of personal choice as to which method is preferred. Both methods require a considerable amount of time and development of laboratory technique. The rotating-sensor method, once the regression analysis and computer programing are developed, still requires substantial time in manually reading data sets for input to the regression analysis. The scheme cannot be automated entirely in that some discretion enters into determining local flow direction with angular position at maximum signal, etc. The x-array technique is time-consuming in that four traverses are required through the boundary layer but, on the other hand, requires very little data reduction, with the stresses coming almost directly from the correlator/sum-difference unit. Both methods measure stresses relative to local flow direction, a variable through the traverse, but a simple transformation will give the stress field relative to the freestream or body coordinate system.

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Symmetric Stiffness Matrix for Incompressible Hyperelastic Materials

Takao Takamatsu* and James A. Stricklin†
Texas A&M University, College Station, Texas
and
John E. Key‡
NASA Marshall Spaceflight Center, Huntsville, Ala.

SOME materials have the characteristics that the stress tensor is derivable from a strain energy function and that the deformation occurs without an appreciable change in volume. These are called incompressible hyperelastic materials, which include the significant difficulty that in plane strain or axisymmetric problems the stress tensor is not determined by only the deformation. A hydrostatic pressure that does not affect the strain tensor must be considered to

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*Research Assistant, Department of Aerospace Engineering.

†Professor, Department of Aerospace Engineering. Member AIAA.

‡Staff Engineer. Member AIAA.

evaluate the stress tensor and is referred to as a Lagrange multiplier.[1-3]

In the case of incompressible hyperelastic materials, the modified strain energy is defined as

$$\bar{W}=W(I_1, I_2)+p(I_3-1) \tag{1}$$

where p is a Lagrange multiplier, I_1, I_2, and I_3 denote the first, second, and third principal strain invariants of Green's deformation tensor, and W is a strain energy function that is taken widely as the Mooney-Rivlin form.

By definition, the stress tensor is calculated using the modified strain energy function (1):

$$S_{ij}=[\partial W(E_{kl})/\partial E_{ij}]+p[\partial I_3(E_{kl})/\partial E_{ij}] \tag{2}$$

in which S_{ij} are components of the second Piola-Kirchhoff stress tensor, and E_{ij} denote components of the Green-Lagrange Strain tensor.

The principle of virtual displacements is used to express equilibrium equations over an arbitrary finite element of a continuous body in a configuration required. System of nonlinear equilibrium equations over the element is obtained:

$$\int_{V_0} S_{ij}\delta E_{ij}\, dV=\delta W^e \tag{3}$$

where V_0 is an undeformed volume of the element, and δW^e is an external virtual work.

Substituting Eq. (2) into Eq. (3), taking the Taylor's expansion at previously calculated configuration $(\bar{E}_{ij}, \bar{S}_{ij}, \bar{p})$, and neglecting nonlinear terms in the unknowns, linear equations in the incremental displacements and Lagrange multiplier are written as

$$\int_{V_0}\Delta e_{rs}C_{ijrs}\delta\Delta e_{ij}dV+\Delta p\int_{V_0}\frac{\partial I_3(\bar{E}_{kl})}{\partial E_{ij}}\delta\Delta e_{ij}dV$$
$$+\int_{V_0}\bar{S}_{ij}\delta\Delta\eta_{ij}dV=\delta W^e-\int_{V_0}\bar{S}_{ij}\delta\Delta e_{ij}dV \tag{4}$$

where Δe_{ij} and $\Delta\eta_{ij}$ are components of linear and nonlinear incremental strain tensor, Δp is an incremental Lagrange multiplier, and

$$C_{ijrs}=[\partial^2 W(\bar{E}_{kl})/\partial E_{ij}\partial E_{rs}]+\bar{p}[\partial^2 I_3(\bar{E}_{kl})/\partial E_{ij}\partial E_{rs}] \tag{5}$$

Since Eq. (4) includes an extra variable Δp, the incompressibility condition is introduced as an additional equation needed. The incompressibility condition assumed is satisfied in an average sense over the element; then we set, for the linearized incompressibility condition,

$$\int_{V_0}\frac{\partial I_3(\bar{E}_{kl})}{\partial E_{ij}}\Delta e_{ij}dV=-\int_{V_0}\{I_3(\bar{E}_{kl})-1\}dV \tag{6}$$

Since the hydrostatic pressure has been assumed constant over the element, a constant strain element is suitable to this problem. Employing the general relationship[4] between incremental strains and nodal displacements for rewriting Eqs. (4) and (6) in a simple matrix form, we easily can find the remarkable characteristic that the stiffness matrix of the element can be formed symmetric:

$$\begin{bmatrix} k_L+k_{NL} & k_I \\ k_I^T & 0 \end{bmatrix}\begin{Bmatrix}\Delta u \\ \Delta p\end{Bmatrix}=\begin{Bmatrix} r-f \\ i\end{Bmatrix} \tag{7}$$

where

$$[k_L]=\int_{V_0}[B_L]^T[C]\,[B_L]dV,$$

$$[k_{NL}]=\int_{V_0}[B_{NL}]^T[\bar{S}]\,[B_{NL}]dV$$

$$\{k_I\}=\int_{V_0}[B_L]^T\{\frac{\partial I_3(\bar{E}_{kl})}{\partial E_{ij}}\}dV,\ \{f\}=\int_{V_0}[B_L]^T\{\bar{S}\}dV$$

$$i=-\int_{V_0}[I_3(\bar{E}_{kl})-1]dV$$

$\{\Delta u\}$ and $\{r\}$ are the column matrices of incremental nodal displacements and nodal forces, $\{\Delta e\}=[B_L]\{\Delta u\}$, and $[B_{NL}]$ and $[\bar{S}]$ used are derived by Bathe et al.[5]

Finally, assembling the structure matrices follows the standard procedure with attention that character of symmetry is not broken, and we obtain the symmetric structure matrices in the following symbolic form:

$$\begin{bmatrix} K_L+K_{NL} & K_I \\ K_I^T & 0 \end{bmatrix}\begin{Bmatrix}\Delta U \\ \Delta P\end{Bmatrix}=\begin{Bmatrix} R-F \\ I\end{Bmatrix} \tag{8}$$

We can note emphatically that the symmetric structure matrices now derived possess the better opportunity (that finite-element programs commonly used are modified simply and employed for solving problems of incompressible hyperelastic materials) than unsymmetric structure matrices previously used.

As an example solved by this symmetric stiffness matrix, an infinite hollow cylinder, 7-in. internal radius and 18.625-in. external radius, of Mooney-Rivlin material with $C_1=80$ psi, $C_2=20$ psi, is considered. Ten constant strain triangular axisymmetric elements are employed, and 10 calculations iterated by the Newton-Raphson technique are taken to obtain the convergent solutions. After each iteration, the generalized forces at the interior nodes due to internal pressure must be increased in proportion to the expansion of internal radius. Displacement and hydrostatic pressure profiles are shown in Fig. 1. The displacement points are the average values of top and bottom nodes, and the hydrostatic pressure points are plotted with the average values of upper and lower elements in a manner similar to Oden and Key.[1] We see that our results are in good agreement with the exact curves. The difference

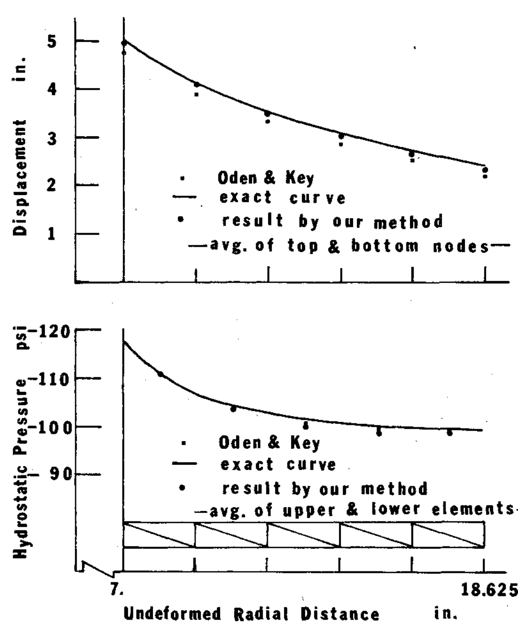

Fig. 1 Displacement and hydrostatic pressure profiles of an infinite hollow cylinder subjected to internal pressure ($C_1=80$ psi, $C_2=20$ psi, internal radius = 7, external radius = 18.625 in., internal pressure = 128.2 psi, 10 elements, 10 Newton-Raphson iterations).

between our and Oden and Key's results may be caused by the distinction of two methods, the Newton-Raphson and the incremental loading used by them.

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Technical Comments

Comment on "Perturbation Method of Structural Design Relevant to Holographic Vibration Analysis"

Alexander H. Flax*
Institute for Defense Analyses, Arlington, Va.

ALTHOUGH Stetson[1] has referred to an original source[2] for Rayleigh's principle, he has apparently overlooked, as many others have, the original and complete treatment of perturbation theory, also given by Rayleigh in that same source. In fact, except for one small, but sometimes significant, difference, Stetson's principal results, namely Eq. (21) for the changes in frequencies and Eq. (30) for the changes in normal modes, correspond exactly to the results originally given by Rayleigh in Ref. 2 for first-order perturbation theory. To see this, it is only necessary to substitute in his Eqs. (21) and (30) for the nth generalized stiffness K_n, the equivalent terms $\omega_n^2 M_n^x$, where ω_n is the natural frequency of the nth normal mode and M_n is the corresponding generalized mass. Rayleigh, in the same work, also gives the complete development of second-order perturbation theory.

The only difference between the usual form of first-order perturbation theory applied to vibration problems and Stetson's results is that, following Rayleigh for undamped systems, it is customary to calculate the perturbation in frequency squared, $\Delta(\omega_n^2)$. Whereas Stetson computes $\Delta\omega_n$ so that $\Delta(\omega^2)$ is given by $2\omega_{no}\Delta\omega_n$ where ω_{no} is the unperturbed natural frequency. Formally, to mathematical terms of the first order, these expressions are equivalent. In practice it appears to be neither necessary nor desirable to make this substitution. For example, if the stiffness of a structure is uniformly increased 10% (say by changing materials), the change in ω_n^2 is exactly 10%. The new values of ω_n are exactly $1.1\omega_{no}$ or 4.88% higher than the original values. However, using Stetson's formulas, we get a change of 5%. If the new value of ω_n is taken as $1.05\omega_{no}$, then its square ω_n^2 rather than $1.1\omega_{no}^2$. Although the differences are minor as long as the perturbation parameters are small, there is no reason to change long-established methods to introduce additional errors, no matter how small. Moreover, in certain instances, taking advantage of the variational properties of some perturbation formulas, they are used when perturbations are not small, and in such cases unnecessary errors should certainly be avoided.

Finally, it is worth nothing with regard to the applications of perturbation theory alluded to by Stetson that Rayleigh[2] has specifically discussed the problem of "tailoring" perturbations so as to produce given changes in the natural frequencies of vibrating systems using the vibrating string as an example. He has also shown how mass distribution to achieve desired frequencies can be calculated on a basis more precise than first-order perturbation theory.

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*President. AIAA Fellow.

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Reply by Author to A.H. Flax

Karl A. Stetson*
United Aircraft Corporation, East Hartford, Conn.

THE comments of A.H. Flax are well taken. I did not reference any prior perturbation analyses because I felt there were simply too many, and none that I knew of were really suited to the task I wished to address. The one by Lord Rayleigh is certainly the closest to what I was seeking; however, it is overly concise, occupying a scant two pages, and the illustrations using the string are too simple because only mass perturbations are used. Without demeaning Rayleigh's work, which is truly monumental and of which the perturbation analysis is but one of hundreds of topics, I would like to state what I found lacking. First, mode shapes are not explicitly used in the formulation. Their influence appears implicitly in two parameters, a_r and c_r, which correspond to modal mass and modal stiffness, and which are never explicitly defined except later in the book in various contexts. Second, he never actually stated that the new mode shape is expressible as a series of the old mode shapes, certainly not mathematically. Third, the physical interpretation of mode shape changes in not discussed at all, that is, Rayleigh confined himself to physical interpretations of frequency changes.

The primary point of my article is supported by Rayleigh's introduction to this topic. He presented perturbation as a method of analyzing a complicated system in terms of a simple one for which an analysis is mathematically known. My point is that the experimental technique of hologram interferometry allows even very complicated structures to be taken as starting points for perturbation analysis.

Finally, with regard to applications of perturbation theory to design, a colleague and I have currently under submission to this journal an article in the inverse of the perturbation process to specify structural changes that will change mode shapes and frequencies deterministically. We have formulated this both from a finite element and a continuum point of view. We look forward to the comments of Dr. Flax on this topic.

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*Senior Research Engineer, Instrumentation Laboratory.